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THE INFLUENCE OF MOMENTUM EXCHANGE ON THE PROPAGATION OF DISTURBANCES IN A MULTI-COMPONENT FLUID

Richard L. Liboff

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Abstract

One-dimensional perturbations of a two-component neutral fluid about absolute Maxwellian equilibrium states (with equal temperatures) are examined. The starting equations are two Boltzmann equations coupled through difference-in-velocity terms. A Fourier-Laplace transform analysis indicates that the resulting modes are all strongly damped. It is concluded that a momentum exchange mechanism alone cannot support the collective sound mode. Asymptotic forms for decay frequency are found for small and large k waves. A thermodynamic estimate is given for the sound speed of a multi-component gas in which the number of degrees of freedom of each component is distinct.

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Wave Propagation in a Multi-Component Neutral Fluid: The Influence of Momentum Exchange

I. Introduction and Summary of Results.

When investigating the fluctuations associated with a multi-component fluid a very popular course of action used to couple the components is to use momentum exchange terms in the equations of motion. This is true both in the microscopic vein [1,2,3], where the formidable collision integrals are replaced with simpler difference-in-velocity flow terms, and it is especially true in the macroscopic approach [4,5] where the individual momentum equations are supplanted with inhomogeneous momentum exchange terms. This latter approach has recently been extended to analysis of partially ionized gases [6-11] (i.e., three-component fluids) with specific regard to problems pertaining to ionospheric propagations.

In all of these problems an important goal is that of recapturing the multi-component sound speed. It is the purpose of this paper to show that any analysis of wave propagation in a multi-component fluid whose only interaction mechanism is one of momentum exchange will exhibit only strongly damped modes, and consequently can not support the proper (undamped in one dimension) multi-component sound phenomenon. This observation, of course, renders the included model unrealistic.

The included starting equations are two one-dimensional Boltzmann equations, which have been linearized about absolute Maxwellian state with equal temperatures. These equations are coupled through difference-in -velocity forcing terms. A Fourier-Laplace transform analysis is applied to a generalized boundary-initial value problem. The results are obtained by examining the poles of the integrand of the inverse transform. These, in turn, indicate, that there are no growing modes* (i.e., the only solutions are for Re $s < 0$, where s is the time transform variable). Analytic continuation of the transform also yields no roots, both for Re $s = 0$ (pure propagation) and in a neighborhood infinitesimally close to the imaginary s axis (propagation with small attenuation). The only solutions that remain are those which are strongly damped. Two specific extremes are calculated. The first pertains to large k (k = wave number) phenomena. These waves decay with a frequency which varies as k^2. The second example pertains to small k phenomena, whose decay frequency varies in a slow logarithmic-like manner with k.

In retrospect, it is clear that a momentum exchange mechanism will not support the sound phenomena. This mechanism

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* This, of course, is equivalent to the fact that there are no "normal" modes.

serves primarily to quench macroscopic velocity differences but cannot force local equilibrium. Such equilibrium is attained in a one-component fluid, for instance, through a BGK-like [12] interaction.

As pointed out above, in studies of wave propagation in partially ionized gases, the mechanism of momentum exchange is used to couple the three components. However, inasmuch as two of the species are charged, there is another coupling mechanism, namely that afforded by Maxwell's equations. This electrical coupling enforces an equilibrium corresponding to the state of charge neutrality. So it is, for instance, that Landau's [13] analysis which includes only long range averaged electric fields, without any short range collisions, gives propagating waves, which in the limit of the long wavelengths severely resemble sound phenomena. However, the true fluid dynamical long wavelength sound mode (e.g., as appears in the Lundquist equations) has been shown to vanish in the limit of vanishing short range collision frequency [3].

A recent macroscopic analysis by Tanenbaum and Mintzer [11] of wave propagation in a partially ionized gas includes both frictional and electrical coupling between the species. In addition, equations of state are included for each of the three components, which in turn insures internal equilibrium for the separate species. Under these conditions the momentum exchange mechanism may well enforce total equilibrium inasmuch as it is driving together components already in equilibrium. Indeed, in

the limit of low frequencies, the adiabatic sound speed, $\sqrt{\gamma p/\rho}$ is recaptured, where p and ρ are equilibrium total pressure and mass density respectively.

A brief thermodynamic derivation of the cooperative sound speed is included in the appendix.

II. Analysis

Our starting equations are two one-dimensional augmented Boltzmann equations [3] for the perturbation distribution f:

(1) $$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = \nu_1 f_o \frac{\mu}{m} \xi \Delta u$$

(2) $$\frac{\partial \hat{f}}{\partial t} + \xi \frac{\partial \hat{f}}{\partial x} = -\nu_1 \hat{f}_o \frac{\mu}{\hat{m}} \xi \Delta u$$

where

(3) $$\Delta u = \hat{u} - u$$

(4) $$\mu = \frac{\hat{m}\hat{n}_o m n_o}{\hat{m}\hat{n}_o + m n_o}$$

(5) $$KT_o = m\, C^2 = \hat{m}\, \hat{C}^2$$

(6) $$f_o = \frac{n_o}{(2\pi)^{\frac{1}{2}}\, C} \exp(-\xi^2/2C^2).$$

Furthermore, the distributions are normalized so that,

$$\int f_o = n_o \tag{7}$$

$$\int f\xi = n_o u. \tag{8}$$

The variables associated with "gas No. 1" are roofed while those of "gas No. 2" are bare. The two gases have the same temperature T_o in the equilibrium state which, in turn, is specified by the absolute Maxwellians f_o and $\hat{f}_o$. The perturbed state away from equilibrium is given by the total distribution,

$$F = f_o + f; \qquad \hat{F} = \hat{f}_o + \hat{f} . \tag{9}$$

Number density is n, macroscopic velocity is u, m is mass, μ is reduced mass,** C is thermal speed, ξ is microscopic speed, and ν is cross-collision frequency.

The problem that is considered is a mixed initial-boundary value problem with* $f(x,0) = h(x)$ and $f(\pm \infty,0) = 0$. The method of solution is that of the Laplace-Fourier transform:

$$f^*(s,k) = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dt \; e^{-st+ikx} f(t,x) \tag{10}$$

$(\mathrm{Re}\ s > 0)$

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* All distributions are functions of velocity ξ.

**More accurately, μ is a density weighted reduced mass.

with corresponding inverse,

$$(11) \qquad f(t,x) = \frac{1}{i(2\pi)^2} \int_{-\infty}^{\infty} dk \int_{\gamma - i\infty}^{\gamma + i\infty} ds \; e^{st-ikx} \, f^*(s,k) \; .$$

Multiplying equations (1,2) through by the integral operator $\int dx \int dt \exp[-st+ikx]$ gives the desired transformed equation,

$$(12) \qquad [s - ik\xi]f^* - h(x) = \nu_1 \frac{\mu}{m} \xi \frac{\Delta u^*}{c^2} f_o$$

$$(13) \qquad [s - ik\xi]\hat{f}^* - \hat{h}(x) = -\nu_1 \frac{\mu}{\hat{m}} \xi \frac{\Delta u^*}{c^2} \hat{f}_o \; .$$

Solving for f^* gives

$$(14) \qquad f^* = \nu_1 \frac{(\mu/m)\, \xi \, (\Delta u^*/c^2) f_o}{s-ik\xi} + \frac{h(x)}{s-ik\xi} \; , \quad .$$

with a similar expression for $\hat{f}^*$. Forming the velocity moment gives

$$(15) \qquad u^* = \frac{\nu_1 \mu}{imkC} [\hat{u}^* - u] \int_{-\infty}^{\infty} \frac{(2\pi)^{-\frac{1}{2}} e^{-v^2/2} \, v^2 dv}{z-v} + \langle h \rangle$$

$$(16) \qquad u^* = \frac{-\nu_1}{i\hat{m}\hat{k}c}\,[\hat{u}^*-u]\int_{-\infty}^{\infty}\frac{(2\pi)^{-\frac{1}{2}}e^{-v^2/2}\;v^2\,dv}{\hat{z}-v} + \langle\hat{h}\rangle$$

$$z = s/iCk$$

where $\langle h\rangle$ is the implied initial distribution moment. Subtraction of these two equations gives one equation for the single unknown Δu^*,

$$(17) \qquad \Delta u^* = \frac{\Delta\,\langle h\rangle}{\frac{1}{iak}\,(CI(z) + \hat{C}I(\hat{z})) + 1}$$

where

$$(18) \qquad a = KT_o/\nu_1\mu$$

$$(19) \qquad I(z) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{v^2e^{-v^2/2}\;dv}{z-v}$$

$$(20) \qquad \langle\Delta h\rangle = C^2\int_{-\infty}^{\infty}\frac{e^{-v^2/2}[\hat{h}-h]v\,dv}{z-v}\;.$$

The inverse of (17) appears as

$$(21)\qquad \Delta u = \frac{1}{i(2\pi)^2}\int_{-\infty}^{\infty} dk e^{-ikx}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{ds e^{st}\Delta\langle h\rangle}{[CI(z)+\hat{C}I(\hat{z})]\frac{1}{iak}+1}\ .$$

In the event of vanishing coupling $a \longrightarrow \infty$, and only the $\Delta\langle h\rangle$ factor remains which, in turn, induces the streaming solution $\Delta u = \Delta u(0)$. More generally, both the poles of $\Delta\langle h\rangle$ and the roots of

$$(22)\qquad CI(z) + \hat{C}I(\hat{z}) = -iak$$

contribute to the time development of Δu. In uncovering the roots of eq.(22) it is important to recall that the definition of f^*, and therefore Δu^*, includes $\mathrm{Re}\ s > 0$, which (for positive k) relates to $\mathrm{Im}\ z < 0$.

Examination of the I functions quickly indicates that there are no roots for $\mathrm{Im}\ z < 0$ which, in turn, relates to the fact that the starting equations (1,2) with the included boundary-initial conditions do not induce growing modes. A separate analysis for z on the real axis indicates that there are also no roots there — so that there are no purely propagating modes. Indeed, continuing the I functions into the upper half z-plane gives no roots in an open region above the real axis — so that there are no propagating modes which are only slightly damped. The only solutions that equation (21) has are in the remaining

section of the upper half z-plane.

In order to exhibit these facts we first rewrite $I(z)$ in the form,

$$(19') \qquad I(z) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{e^{-v^2/2}\, v^2(\bar{z}-v)dv}{|z-v|^2} \qquad (\mathrm{Im}\ z < 0),$$

so that

$$(23) \qquad \mathrm{Im}\ I(z) = D\ \mathrm{Im}\ \bar{z} = -Dy < 0 \qquad (y > 0),$$

Where D is the implied real positive definite function.

For z approaching the real axis the contour must be distorted as depicted in Fig. 1.

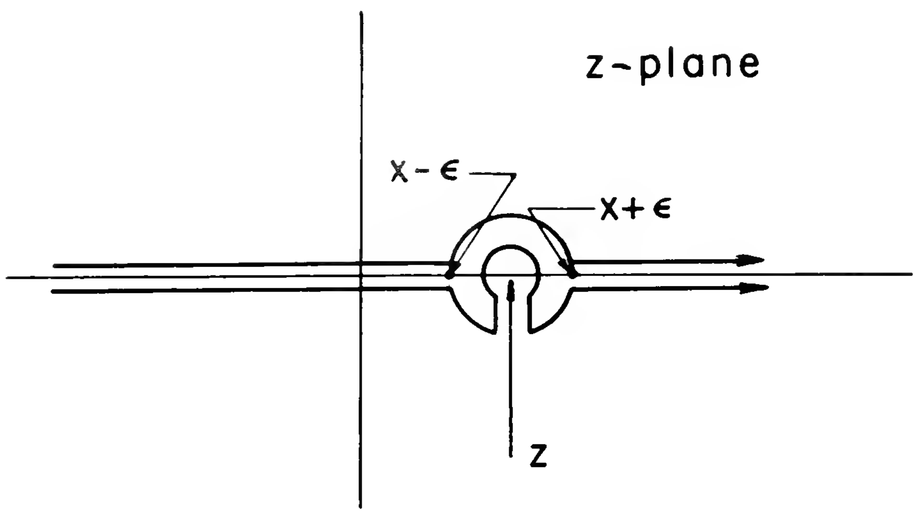

Figure I. Continuation onto the Real Axis.

In the limit as $\varepsilon \longrightarrow 0$, one obtains the proper analytic combination of $I(z)$ into the real axis [13], which appear as,

$$(24) \qquad I(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \, P\!:\!\int \frac{e^{-v^2/2}\, v^2 dv}{x-v} + i(\pi/2)^{\frac{1}{2}}\, x^2 e^{-x^2/2}, \qquad (\mathrm{Im}\ z = 0)$$

where the P: symbol indicates that the principal part is to be taken. It follows that along the real axis,

$$(25) \qquad \mathrm{Im}\ I(x) = (\pi/2)^{\frac{1}{2}}\, x^2\, e^{-x^2/2},$$

a positive definite form.

In the upper half plane the contour is distorted as depicted in Figure 2,

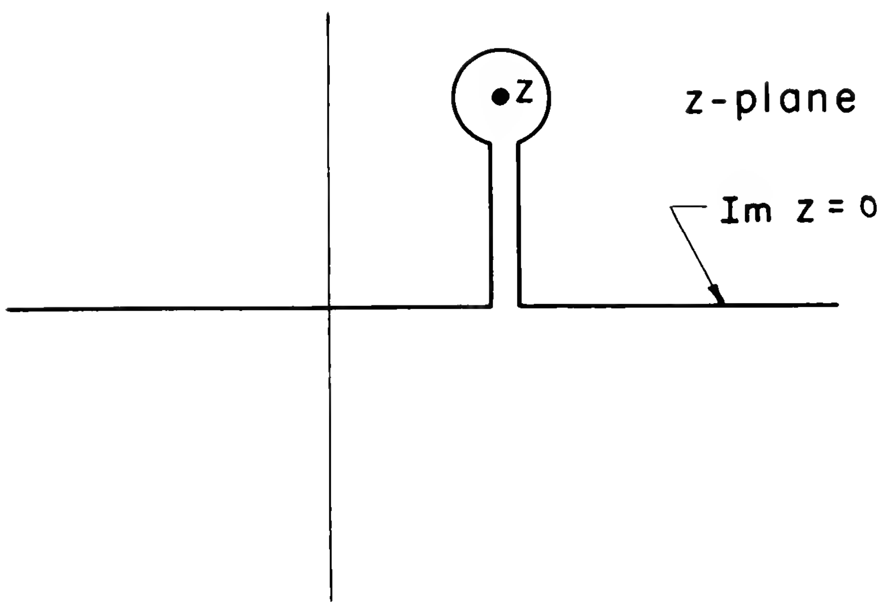

Figure 2. Continuation into the Upper Half Plane.

to obtain

$$(26) \qquad I(z) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{e^{-v^2/2}\, v^2 dv}{z-v} + i(\pi/2)^{\frac{1}{2}}\, z^2 e^{-z^2/2} \qquad (\mathrm{Im}\ z > 0),$$

or equivalently,

$$(27) \qquad I(x+iy) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{(\bar{z}-v)v^2\, e^{-v^2/2}\, dv}{(x-v)+iy}$$

$$+ i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} (x^2-y^2+2ixy)\, e^{-[x^2-y^2+2ixy]^{\frac{1}{2}}},$$

and also

$$(28) \qquad I(z) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{(\bar{z}-v)v^2\, e^{-v^2/2}\, dv}{|z-v|^2} + i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} z^2 e^{-z^2/2}.$$

a. <u>Growing Modes</u>

We are now prepared to exhibit formally the statements made above. First, for $\mathrm{Im}\ z < 0$, comparison of equations (21-23) immediately reveals that there can be no solutions in this domain, granted that $k > 0$. This, of course, implies that there are similarly no growing modes for $k < 0$, inasmuch as the starting equations are invariant under reflexion of coordinates.

However, it is also evident if we merely set $k \longrightarrow k' = -k$, so that $z \longrightarrow z' = -z$, and $\operatorname{Im} I(z)$ becomes positive, and equation (22) again has no solutions in the domain of growing modes.

b. Purely Propagating Modes

The purely propagating waves correspond to roots along the real z-axis. Comparison of the dispersion equation (21) and equation (24) again shows that there are no solutions for $k > 0$. The above space reversal argument shows that there are also no solutions for $k < 0$. More explicitly, consider again that $k \longrightarrow k' = -k$, so that $z \longrightarrow z' = -z$. Inasmuch as $I(z)$ is defined primarily for $\operatorname{Re} s > 0$, the real axis must now be approached from above and the Cauchy contribution reverses sign since the direction of the loop integral is reversed. It follows that the imaginary parts of both the right and left sides of (21) change sign, and solution is still impossible along the real axis.

c. Slightly Damped Modes

These modes are those lying in the region of the z-plane depicted in Figure 3. The proof that there are no slightly damped modes is given first for the ε-wedged region, and then for the region about the origin.

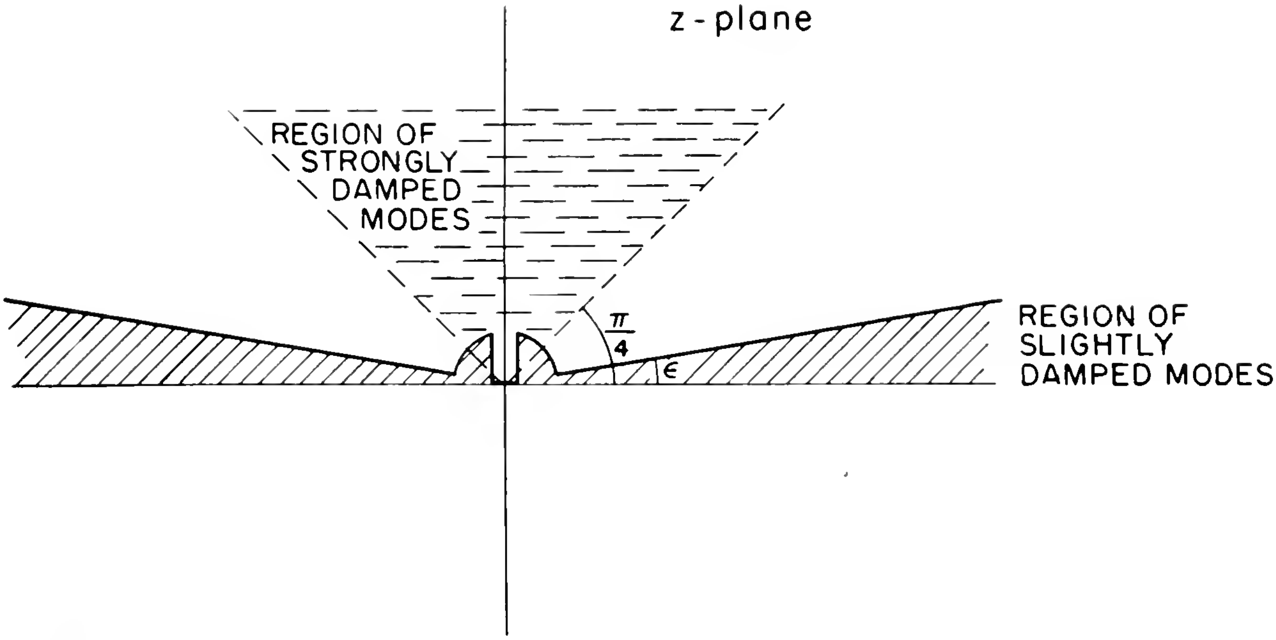

Figure 3. **Regions of Attenuation**

The wedged region is given by $0 < \theta \leq \varepsilon$ and $\pi - \varepsilon \leq \theta < \pi$. In that region, [illegible] and $\cos \varepsilon = O(1)$. Substituting $z = re^{i\theta}$ into eq. (28) and retaining terms to order unity gives*

$$\text{(29)} \qquad \operatorname{Im} I(z) = E(\cos [illegible]\varepsilon)e^{-r^2\cos^2\varepsilon} + O(\varepsilon).$$

---

* More precisely, the strongly damped modes lie in a cone about the imaginary z-axis with apex $\pi/$[illegible]. The estimate given in Eq.(29) begins to fail drastically near $\varepsilon = \pi/4$, where $\cos \varepsilon \approx \sin \varepsilon$.

(where B is real and positive) which, in turn, indicates that the imaginary part of the dispersion relation equation (22) cannot be satisfied in this domain.

In the remaining domain about the origin (excluding the imaginary axis), if $z = \varepsilon$ is substituted into eq. (26) then one obtains,

$$\text{(30)} \qquad I(z) = -\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} v\, e^{-v^2/2} dv [1 + (\frac{z}{v}) + (\frac{z}{v})^2 + \dots]$$

$$+ i(\frac{\pi}{2})^{\frac{1}{2}} z^2 (1 - \frac{z^2}{2} + \dots),$$

so that to terms of $O(z^2)$,

$$\text{(31)} \qquad I(z) \sim -z + i(\frac{\pi}{2}) z^2 + O(z^3).$$

Consider now the real part of the left side of the dispersion equation (21). To terms of $O(z)$, there results

$$\text{(32)} \qquad \mathrm{Re}[CI(z) + \hat{C}I(\hat{z})] = -[Cx + \hat{C}\hat{x}],$$

which cannot be zero for $x > 0$, so that eq. (22) has no solutions in this domain.

d. Remaining Solutions

The above analysis indicates that the only solutions of the dispersion equation (22) lie in the sector of the upper half z-plane shown in Fig. 4.

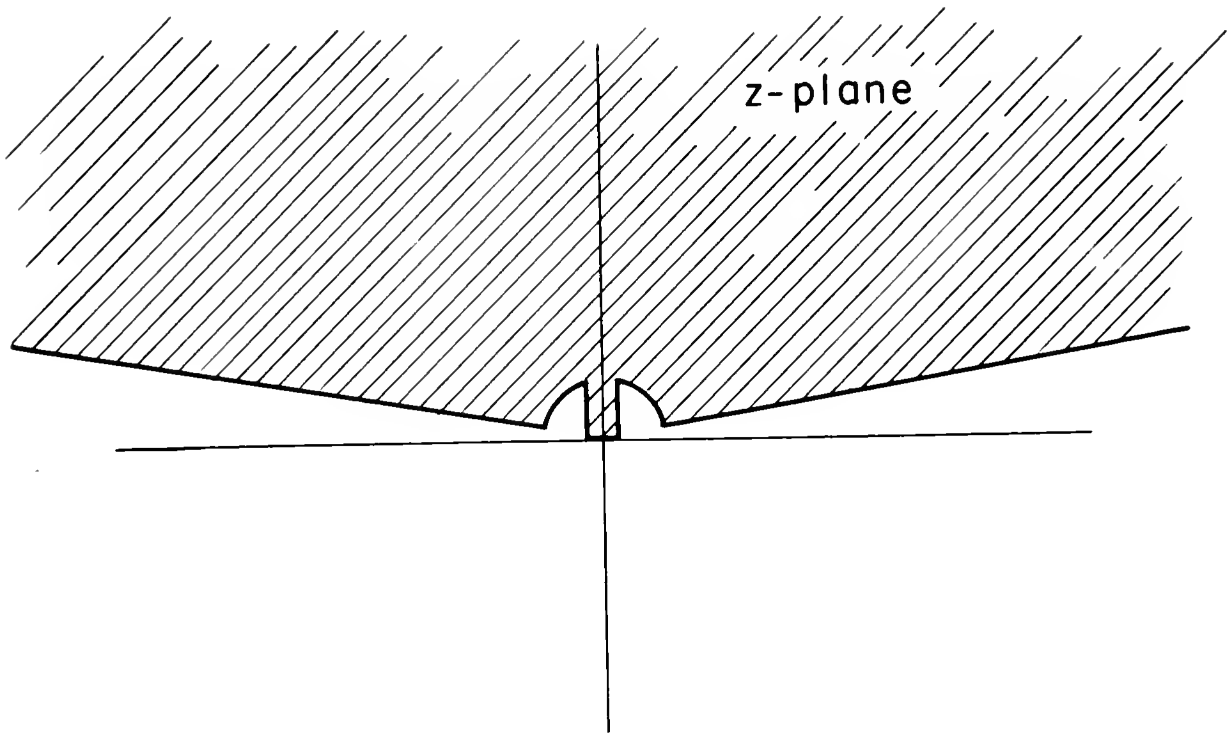

Figure 4. Domain of Solution

That there are any solutions at all follows from the entire nature of the I functions about the point at infinity. That I is entire may be seen by relating it to another

entire function, viz., the Landau [13] function L,

(33) $$L = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{v\ e^{-v^2/2}\ dv}{v-z} .$$

This is related to the I integral through

(34) $$-I = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{v(v-z+z)e^{v^2/2}\ dv}{v-z} = zL$$

so that I is entire, granted that L is entire*.

We will now exhibit two explicit solutions in this remaining domain, both of which lie along the imaginary axis.

First for $z = iy$, $y << 1$, $[k >> s/C]$

the form (26) gives

(35) $$I(iy) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int \frac{e^{-v^2/2}\ v^2 dv}{iy-v} - i(\frac{\pi}{2})^{\frac{1}{2}}\ y^2\ e^{y^2/2}$$

$$= - \frac{iy}{(2\pi)^{\frac{1}{2}}} \int \frac{e^{-v^2/2}\ v^2}{y^2+v^2} - i(\frac{\pi}{2})^{\frac{1}{2}}\ y^2 e^{y^2/2} \sim - iy+O(y^2).$$

---

* Indeed, a disperson equation identical to (22), only in terms of L rather than I, results if the difference between the two continuity equations are combined with the number density moments of f (instead of velocity moments) to again obtain one equation for Δu.

If this, in turn, is substituted into (21) there results,

$$\text{(36)} \qquad - i[Cy + \hat{C}\hat{y}] = + \frac{i2s}{k} = - \frac{ikKT_o}{\nu_1\mu}$$

where we have recalled that $z = iy = s/ick$, so that $y = -s/Ck$. The root of (36) appears as

$$\text{(37)} \qquad s = - k^2 \frac{KT_o}{2\nu_1\mu} .$$

Our second example also involves roots along the imaginary axis $z = iy$; however, in this case $y >> 1$. The Cauchy contribution dominates over the integral part in this region so that (26) yields

$$\text{(38)} \qquad I(z) \sim - i(\frac{\pi}{2})^{\frac{1}{2}} y^2 e^{y^2/2}.$$

The dispersion equation (22) now reads,

$$\text{(39)} \quad G(y) = (\frac{\pi}{2})^{\frac{1}{2}} \left\{ y^2 c\ e^{y^2/2} + \hat{y}^2\hat{C}\ e^{\hat{y}^2/2} \right\} = ka = - \frac{s}{y} (\frac{a}{C}).$$

Since $\hat{y} = y[\hat{c}/c]$, the left side of this equation may be written purely as a function of y, viz., $G(y)$. This function is more rapidly increasing than exponental (for large y). However, to gain a rough estimate of the behavior of $s = s(k)$, let us suppose that $G(y) \sim e^{y^2}$, so that $y \sim \ell n(ka)^{\frac{1}{2}}$ and $s \sim ka\ \ell n(ka)^{\frac{1}{2}}$.

The actual G is a more rapidly increasing function of y so that, in truth, s varies even more slowly than this logarithmic estimate indicates.

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Appendix.

The formula for the sound speed of a multi-component gas, which follows from fluid dynamics [14], is

$$a^2 = \frac{\partial p}{\partial \rho}, \tag{A.1}$$

evaluated at constant entropy and at equilibrium. The total pressure p and total mass density ρ are related to the component values p_i and ρ_i, respectively, through

$$p = \Sigma\, p_i \tag{A.2}$$

$$\rho = \Sigma\, \rho_i . \tag{A.3}$$

A similar relation holds between the total internal energy E and the component energies E_i, viz.,

$$E = \Sigma\, E_i . \tag{A.4}$$

The equilibrium state of each component obeys the relations,

$$E_i = \frac{f^i}{2}\, \rho_i R_i T_i V = \frac{f^i}{2}\, p_i V, \tag{A.5}$$

$$p_i = \rho_i R_i T_i , \tag{A.6}$$

$$T_i = T_o . \tag{A.7}$$

The mass density ρ_i is related to the total volume through the elemental mass m_i and the number of particles in the i^{th} species N_i according to

$$\rho_i = \frac{m_i N_i}{V} = \frac{c_i}{V}, \tag{A.8}$$

$$\rho = \Sigma\rho_i = \frac{1}{V}\,\Sigma c_i = \frac{c}{V}, \tag{A.9}$$

where c is the implied constant. The number of degrees of freedom of an element of the i^{th} species is f^i. The constant R_i is K/m_i where K is Boltzmann's constant.

In order to calculate $\partial p/\partial\rho$ at constant entropy S, we employ the first law of thermodynamics, which under the said constraint appears as,

$$Tds = 0 = dE + pdV. \tag{A.10}$$

Substituting the above equilibrium relations gives,

$$\frac{dp}{p} = -\left\{\left(\frac{c}{dp}\,\Sigma\, f^i p_i\right)\Bigg/\left(\frac{\Sigma\, p_i(f^i+2)}{p}\right)\right\}\frac{dV}{V}. \tag{A.11}$$

If we assume that the term in brackets is slowly varying in the adiabatic variation (A.10), then (A.11) is readily integrated to obtain the familiar form,

$$p = A\rho^{\overline{\gamma}}, \tag{A.12}$$

where we have set $\bar{\gamma}$ equal to the bracketed term in (A.11), and eliminated V through equation (A.9). The constant A is as implied. Equation (A.12) yields the desired form:

$$a^2 = \bar{\gamma}(p/\rho). \tag{A.13}$$

If all the f^i are equal then the sound speed becomes

$$[\frac{f+2}{f}] \frac{\Sigma p_i}{\Sigma \rho_i} = (\frac{f+2}{f}) \, KT_o \left(\frac{\Sigma n_i}{\Sigma n_i m_i} \right), \tag{A.14}$$

where n^i is the number density of the i^{th} species.

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